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LINEAR ALGEBRA AND ITS APPLICATIONS

Convex combinations of matrices – full rank characterization

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Abstract

Let $\mathbf{A}_1, \ldots, \mathbf{A}_k$ be complex n-by-m matrices. Using the notion of a block P-matrix, introduced recently by Elsner and the second author, a characterization of the full rank property of all convex combinations of $\mathbf{A}_1, \ldots, \mathbf{A}_k$ is derived.

1. Introduction

The purpose of this paper is to characterize the full rank property of all convex combinations of n-by-m complex matrices $\mathbf{A}_1, \ldots, \mathbf{A}_k$.

A motivation to consider such a problem comes from the studies of solving underdetermined systems of linear algebraic equations ([1], Ch. 5.7), from the studies of solving the full rank least squares problem ([1], Ch. 5.3), and from the studies of linear systems of ordinary differential equations with regard to the topics related to controllability ([2], Appendix 2), ([3], Ch. 2.4). We also refer to the role of a full rank property in statistical analysis ([4], Ch. 4) and in research on the block-eigenvalue problem [5]. The full rank property in the case $k = 2$ has been characterized in [6]. To provide a characterization in the general

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case we use the notion of a block P-matrix. To define it we need some notation. By $N(k)$ we will denote a fixed partition of the set $N = \{1, \dots, n\}$ into k, $1 \leqslant k \leqslant n$, pair-wise disjoint nonvoid subsets N_j of the cardinality n_j. $\mathscr{T}^{(k)}$ is then the set of all diagonal n-by-n matrices $\mathbf{T}$ such that, for $i = 1, \dots, k, \mathbf{T}[N_i] = t_i \mathbf{I}_{n_i}$, where $\mathbf{T}[N_i]$ is the principal submatrix of $\mathbf{T}$ with row and column indices in $N_i, t_i \in [0, 1]$ and $\mathbf{I}_{n_i}$ is the n_i-by-n_i identity matrix (we set $\mathbf{I}_n = \mathbf{I}$). Now we are able to define block P-matrices.

Definition [7]. Let $N(k)$ be a partition of N. An n-by-n complex matrix $\mathbf{B}$ is called a block P-matrix with respect to the partition $N(k)$, if for any $\mathbf{T} \in \mathscr{T}^{(k)}$

$$\det(\mathbf{TB} + (\mathbf{I} - \mathbf{T})) \neq 0.$$

2. Preliminaries

Throughout this paper it will be assumed that $\mathbf{A}_i$, $i = 1, \dots, k$, are n-by-m complex matrices and that $k \geqslant 2$. Furthermore, by $\mathbf{\Theta}_k$ we will denote the k-by-k null matrix (we will set $\mathbf{\Theta}_n = \mathbf{\Theta}$).

To formulate our results we introduce some definitions. For given integer k and $\mathbf{A}_1, \dots, \mathbf{A}_k$ we define the n-by-n matrices $\mathbf{A}_{ij}$, $1 \leqslant i \leqslant j \leqslant k$, by

$$\begin{aligned} \mathbf{A}_{ii} &= \mathbf{A}_i \mathbf{A}_i^*, \quad i = 1, \dots, k, \\ \mathbf{A}_{ij} &= \mathbf{A}_i \mathbf{A}_j^* + \mathbf{A}_j \mathbf{A}_i^* - \mathbf{A}_i \mathbf{A}_i^* - \mathbf{A}_j \mathbf{A}_j^*, \ \text{for } i < j, \end{aligned}$$

where X^* is the Hermitian adjoint of a matrix X.

Observe that, following the definitions of $\mathbf{A}_{ii}$ and $\mathbf{A}_{ij}$, some manipulations will give

$$\left(\sum_{i=1}^{k} \alpha_i \mathbf{A}_i\right)\left(\sum_{i=1}^{k} \alpha_i \mathbf{A}_i\right)^* = \sum_{i=1}^{k} \alpha_i \mathbf{A}_{ii} + \sum_{1 \leqslant i < j \leqslant k} \alpha_i \alpha_j \mathbf{A}_{ij}, \tag{1}$$

where $\alpha_i \in [0, 1], i = 1, \dots, k$, and $\sum_{i=1}^{k} \alpha_i = 1$.

For given integer k and $\mathbf{A}_{ij}$, $1 \leqslant i \leqslant j \leqslant k$, we define the block kn-by-kn matrices $\mathbf{B}_i = (\mathbf{B}_{sr}^i), i = 1, \dots, k, 1 \leqslant s, r \leqslant k$, by

$$\mathbf{B}_{sr}^i = \begin{cases} \mathbf{A}_{ii} & \text{if } (s, r) = (1, 1), \\ \mathbf{A}_{r-1,i} & \text{if } (s, r) \in \{(1, 2), (1, 3), \dots, (1, i)\}, \\ \mathbf{I} & \text{if } (s, r) \in \{(2, 2), (3, 3), \dots, (k, k)\}, \\ -\mathbf{I} & \text{if } (s, r) = (i + 1, 1), \\ \mathbf{\Theta} & \text{otherwise.} \end{cases}$$

So, in particular

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{\Theta} & \cdots & \mathbf{\Theta} \\ -\mathbf{I} & \mathbf{I} & \cdots & \mathbf{\Theta} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{\Theta} & \cdots & \mathbf{\Theta} & \mathbf{I} \end{bmatrix}, \qquad \mathbf{B}_2 = \begin{bmatrix} \mathbf{A}_{22} & \mathbf{A}_{12} & \mathbf{\Theta} & \cdots & \mathbf{\Theta} \\ \mathbf{\Theta} & \mathbf{I} & \mathbf{\Theta} & \cdots & \mathbf{\Theta} \\ -\mathbf{I} & \mathbf{\Theta} & \mathbf{I} & \cdots & \mathbf{\Theta} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{\Theta} & \mathbf{\Theta} & \cdots & \mathbf{\Theta} & \mathbf{I} \end{bmatrix},$$

$$\mathbf{B}_k = \begin{bmatrix} \mathbf{A}_{kk} & \mathbf{A}_{1k} & \cdots & \mathbf{A}_{k-1,k} \\ \mathbf{\Theta} & \mathbf{I} & \cdots & \mathbf{\Theta} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{\Theta} & \cdots & \mathbf{\Theta} & \mathbf{I} \end{bmatrix}.$$

We close this section by recalling two results from [8] which we will need for later use.

Lemma 1. *Let α_i, $i = 1, \ldots, k$, be any numbers with $\sum_{i=1}^{k} \alpha_i = 1$. Then the following are equivalent.*

(i) $\sum_{i=1}^{k} \alpha_i \mathbf{A}_{ii} + \sum_{1 \leqslant i<j \leqslant k} \alpha_i \alpha_j \mathbf{A}_{ij}$ *is nonsingular.*

(ii) $\sum_{i=1}^{k} \alpha_i \mathbf{B}_i$ *is nonsingular.*

Theorem 1. *Let $\mathbf{C}_i$, $i = 1, \ldots, k$, be n-by-n complex matrices. The following are equivalent.*

(i) *All convex combinations of $\mathbf{C}_1, \ldots, \mathbf{C}_k$ are nonsingular.*

(ii) *$\mathbf{C}_k$ is nonsingular and the $(k-1)n$-by-$(k-1)n$ matrix*

$$\begin{bmatrix} \mathbf{C}_1\mathbf{C}_k^{-1} & (\mathbf{C}_2 - \mathbf{C}_1)\mathbf{C}_k^{-1} & (\mathbf{C}_3 - \mathbf{C}_2)\mathbf{C}_k^{-1} & \cdots & (\mathbf{C}_{k-1} - \mathbf{C}_{k-2})\mathbf{C}_k^{-1} \\ -\mathbf{I} & \mathbf{I} & \mathbf{\Theta} & \cdots & \mathbf{\Theta} \\ \mathbf{\Theta} & -\mathbf{I} & \mathbf{I} & \cdots & \mathbf{\Theta} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{\Theta} & \cdots & \mathbf{\Theta} & -\mathbf{I} & \mathbf{I} \end{bmatrix}$$

is a block P-matrix with respect to the partition $\{M_1, \ldots, M_{k-1}\}$ of $\{1, \ldots, (k-1)n\}$, where $M_i = \{(i-1)n + 1, \ldots, in\}$, $i = 1, \ldots, k-1$.

3. Results

Theorem 2. *The following are equivalent.*

(i) *All convex combinations of $\mathbf{A}_1, \ldots, \mathbf{A}_k$ have full row rank.*

(ii) *$\mathbf{A}_k$ has full row rank and the $(k-1)kn$-by-$(k-1)kn$ matrix*

$$\begin{bmatrix} \mathbf{B}_1\mathbf{B}_k^{-1} & (\mathbf{B}_2-\mathbf{B}_1)\mathbf{B}_k^{-1} & (\mathbf{B}_3-\mathbf{B}_2)\mathbf{B}_k^{-1} & \cdots & (\mathbf{B}_{k-1}-\mathbf{B}_{k-2})\mathbf{B}_k^{-1} \\ -\mathbf{I}_{kn} & \mathbf{I}_{kn} & \mathbf{\Theta}_{kn} & \cdots & \mathbf{\Theta}_{kn} \\ \mathbf{\Theta}_{kn} & -\mathbf{I}_{kn} & \mathbf{I}_{kn} & \cdots & \mathbf{\Theta}_{kn} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{\Theta}_{kn} & \cdots & \mathbf{\Theta}_{kn} & -\mathbf{I}_{kn} & \mathbf{I}_{kn} \end{bmatrix}$$

is a block P-matrix with respect to the partition $\{\tilde{M}_1,\ldots,\tilde{M}_{k-1}\}$ *of* $\{1,\ldots,(k-1)kn\}$, *with* $\tilde{M}_i=\{(i-1)kn+1,\ldots,ikn\}$, $i=1,\ldots,k-1$.

(iii) *All convex combinations of* $\mathbf{B}_1,\ldots,\mathbf{B}_k$ *are nonsingular.*

Proof. We will show (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii).

First we proof that (ii) $\Rightarrow$ (iii). As $\mathbf{A}_k$ has full row rank hence $\mathbf{A}_{kk}$ is nonsingular. So, $\mathbf{B}_k$ is nonsingular and the implication in question follows by applying Theorem 1.

Now we show the implication (iii) $\Rightarrow$ (i). From (iii), by Lemma 1 and Eq. (1), we have that for all nonnegative α_i, $i=1,\ldots,k$, with $\sum_{i=1}^k \alpha_i = 1$ the matrix

$$\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right)\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right)^*$$

is nonsingular. So,

$$\operatorname{rank}\left(\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right)\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right)^*\right) = n.$$

But then using results on rank [9] we get

$$\operatorname{rank}\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right) = n$$

for all nonnegative α_i, $i=1,\ldots,k$, with $\sum_{i=1}^k \alpha_i = 1$. So, (iii) $\Rightarrow$ (i) holds.

(i)$\Rightarrow$ (ii): First observe that, by (i), full row rank of $\mathbf{A}_k$ is obvious. Using again (i) and the mentioned results on rank, we get

$$\operatorname{rank}\left(\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right)\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right)^*\right) = n.$$

So, the matrix

$$\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right)\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right)^*$$

is nonsingular. Then, by Eq. (1) and Lemma 1, all convex combinations of $\mathbf{B}_1, \ldots, \mathbf{B}_k$ are nonsingular. The implication in question follows by applying Theorem 1. □

Example 1. Let

$$\mathbf{A}_1 = \begin{bmatrix} 2 & 1.5 & 1 \\ 1 & 3 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2.5 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}.$$

As $\mathbf{A}_3$ is of the full row rank there exists $\mathbf{B}_3^{-1}$ and we can form the 12-by-12 matrix

$$\mathscr{A} = \begin{bmatrix} \mathbf{B}_1\mathbf{B}_3^{-1} & (\mathbf{B}_2 - \mathbf{B}_1)\mathbf{B}_3^{-1} \\ -\mathbf{I}_6 & \mathbf{I}_6 \end{bmatrix},$$

where

$$\mathbf{B}_1 = \begin{bmatrix} 7.25 & 7.50 & 0 & 0 & 0 & 0 \\ 7.50 & 11 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} 6 & 5.50 & -0.25 & -0.25 & 0 & 0 \\ 5.50 & 8.25 & -0.25 & -0.25 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{B}_3^{-1} = \begin{bmatrix} 0.700 & -0.400 & 0.175 & -0.400 & 0 & -0.500 \\ -0.400 & 0.300 & -0.100 & 0.300 & 0 & 0.375 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

To check if $\mathscr{A}$ satisfies the condition (ii) of Theorem 2 we observe that, following Theorem 3.8 from [10], a real P-matrix is a block P-matrix with respect to any partition of the index set. A direct calculation shows that all principal minors of $\mathscr{A}$ are positive, therefore $\mathscr{A}$ is a block P-matrix with respect to the partition $\{\{1,\dots,6\},\{7,\dots,12\}\}$ of $\{1,\dots,12\}$. Hence, by Theorem 2, all convex combinations of $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ are of the full row rank.

Remark 1. In [7], another sufficient condition and some necessary conditions for a matrix to be a block P-matrix with respect to a given partition of the index set are offered. Unfortunately, a numerically verifiable characterization of this notion remains unknown.

Remark 2. In (ii) of Theorem 2 one can assume that for some i, $1 \leqslant i \leqslant k$, $\mathbf{A}_i$ has full row rank and the matrix in (ii) is changed accordingly. One has just to rename the $\mathbf{A}_i$'s.

Remark 3. Comparing Theorem 2 with Theorem 1 it is natural to ask if the matrix in (ii) of the former result can be replaced by the $(k-1)n$-by-$(k-1)n$ matrix

$$\begin{bmatrix} \mathbf{A}_1\mathbf{A}_k^+ & (\mathbf{A}_2-\mathbf{A}_1)\mathbf{A}_k^+ & (\mathbf{A}_3-\mathbf{A}_2)\mathbf{A}_k^+ & \cdots & (\mathbf{A}_{k-1}-\mathbf{A}_{k-2})\mathbf{A}_k^+ \\ -\mathbf{I} & \mathbf{I} & \boldsymbol{\Theta} & \cdots & \boldsymbol{\Theta} \\ \boldsymbol{\Theta} & -\mathbf{I} & \mathbf{I} & \cdots & \boldsymbol{\Theta} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{\Theta} & \cdots & \boldsymbol{\Theta} & -\mathbf{I} & \mathbf{I} \end{bmatrix}, \tag{2}$$

where $\mathbf{A}_k^+$ is the Moore–Penrose generalized inverse of $\mathbf{A}_k$. The answer is negative as shown in the following example.

Example 2. Let $\mathbf{A}_i = \left(a_j^{(i)}\right)$, $i = 1,\dots,k-1$, be 1-by-k matrices with

$$a_j^{(i)} = \begin{cases} 1 & \text{if } j \leqslant i, \\ 0 & \text{otherwise} \end{cases}$$

and let $\mathbf{A}_k = \left(a_j^{(k)}\right)$ be a 1-by-k matrix with

$$a_j^{(k)} = \begin{cases} -\varepsilon & \text{if } j = 1, \\ 1 & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

where ε is a positive number.

Then it is easy to see that for any nonnegative numbers α_i, $i = 1,\dots,k$, with $\sum_{i=1}^k \alpha_i = 1$,

$$\operatorname{rank}\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right) = 1.$$

On the other hand, as $\mathbf{A}_k^+$ is the k-by-1 matrix of the form

$$\mathbf{A}_k^+ = \frac{1}{1+\varepsilon^2}\begin{bmatrix} -\varepsilon \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

one can verify that the matrix (2) becomes the $(k-1)$-by-$(k-1)$ lower triangular matrix $L = (l_{ij}),\ 1 \leqslant i, j \leqslant k-1$ with

$$l_{ii} = \begin{cases} \frac{-\varepsilon}{1+\varepsilon^2} & \text{if } i = 1, \\ 1 & \text{otherwise.} \end{cases}$$

So, L is not a block P-matrix with respect to any partition of $\{1, \ldots, k-1\}$.

Concerning Remark 3 we have the following result.

Theorem 3. *Let* $\operatorname{rank}(\mathbf{A}_k) = n$ *and let the* $(k-1)n$*-by-*$(k-1)n$ *matrix* (2) *be a block P-matrix with respect to the partition* $\{M_1, \ldots, M_{k-1}\}$ *of* $\{1, \ldots, (k-1)n\}$ *with* M_i, $i = 1, \ldots, k-1$, *defined in Theorem* 1. *Then all convex combinations of* $\mathbf{A}_1, \ldots, \mathbf{A}_k$ *have full row rank.*

Proof. Since $\operatorname{rank}(\mathbf{A}_k) = n, \mathbf{A}_{kk}$ is nonsingular and $\mathbf{A}_k^+ = \mathbf{A}_k^*\mathbf{A}_{kk}^{-1}$. Hence the matrix (2) becomes

$$\begin{bmatrix} \mathbf{A}_1\mathbf{A}_k^*\mathbf{A}_{kk}^{-1} & (\mathbf{A}_2\mathbf{A}_k^* - \mathbf{A}_1\mathbf{A}_k^*)\mathbf{A}_{kk}^{-1} & (\mathbf{A}_3\mathbf{A}_k^* - \mathbf{A}_2\mathbf{A}_k^*)\mathbf{A}_{kk}^{-1} & \cdots & (\mathbf{A}_{k-1}\mathbf{A}_k^* - \mathbf{A}_{k-2}\mathbf{A}_k^*)\mathbf{A}_{kk}^{-1} \\ -\mathbf{I} & \mathbf{I} & \boldsymbol{\Theta} & \cdots & \boldsymbol{\Theta} \\ \boldsymbol{\Theta} & -\mathbf{I} & \mathbf{I} & \cdots & \boldsymbol{\Theta} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{\Theta} & \cdots & \boldsymbol{\Theta} & -\mathbf{I} & \mathbf{I} \end{bmatrix}.$$

Then, by Theorem 1, all convex combinations of $\mathbf{A}_1\mathbf{A}_k^*, \ldots, \mathbf{A}_k\mathbf{A}_k^*$ are nonsingular. So, we have

$$n = \operatorname{rank}\left(\sum_{i=1}^{k} \alpha_i \mathbf{A}_i \mathbf{A}_k^*\right) = \operatorname{rank}\left(\left(\sum_{i=1}^{k} \alpha_i \mathbf{A}_i\right)\mathbf{A}_k^*\right). \tag{3}$$

Keeping in mind that $\operatorname{rank}(\mathbf{A}_k) = n$ from Eq. (3) it follows that $\operatorname{rank}\left(\sum_{i=1}^k \alpha_i \mathbf{A}_i\right) = n$ for all nonnegative $\alpha_i,\ i = 1, \ldots, k$, with $\sum_{i=1}^k \alpha_i = 1$. So, the proof is completed. □

Remark 4. In Theorem 3 one can assume that for some i, $1 \leqslant i \leqslant k$, $\operatorname{rank}(\mathbf{A}_i) = n$ and the matrix (2) is changed accordingly. One has just to rename the $\mathbf{A}_i$'s.

Remark 5. Theorems 2 and 3 can be applied to the transposes of matrices to get conclusions on the column rank.

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References

[1] G.H. Golub, C.F. Van Loan, Matrix Computations, 3rd ed., Johns Hopkins University Press, Baltimore, 1996.
[2] S. Barnett, Matrices in Control Theory, Van Nostrand Reinhold, London, 1971.
[3] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1991.
[4] C.R. Rao, Linear Statistical Inference and its Applications, Wiley, New York, 1973.
[5] J. Vitoria, Block eigenvalues of block compound matrices, Linear Algebra Appl. 47 (1982) 23–34.
[6] T. Szulc, Rank of convex combinations of matrices, Reliable Computing 2 (2) (1996) 181–185.
[7] L. Elsner, T. Szulc, Convex sets of nonsingular and block P-matrices, Linear and Multilinear Algebra 44 (1998) 1–12.
[8] L. Elsner, T. Szulc, Convex combinations of matrices – nonsingularity and Schur stability characterizations, Linear and Multilinear Algebra (to appear).
[9] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[10] C.R. Johnson, M. Tsatsomeros, Convex sets of nonsingular and P-matrices, Linear and Multilinear Algebra 38 (1995) 233–240.